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A Note on Limiting Properties of Some Bernstein-Type Operators

JESÚS DE LA CAL* AND FRANCISCO LUQUIN*

Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

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Some well-known Bernstein-type operators are exhibited as limits, in an appropriate sense, of other ones. This is readily made by using limit theorems of probability theory. Moreover, in two cases, rates of convergence are also obtained by using probabilistic tools. © 1992 Academic Press, Inc.

1. INTRODUCTION

Since the pioneering work of Bernstein, probabilistic methods have been used with success in dealing with problems of approximation in connection with positive linear operators. The aim of this paper is to contribute in this subject.

To fix ideas, let I be an interval of the real line and let Z^x, Z_1^x, Z_2^x, ..., I-valued random variables whose probability distributions depend upon the parameter $x \in I$. If T and T_n are the positive linear operators respectively associated with Z^x and Z_n^x by means of

$$T(f, x) = Ef(Z^x), \qquad T_n(f, x) = Ef(Z_n^x), f \in CB(I),\ x \in I,$$

where E denotes mathematical expectation and $CB(I)$ stands for the space of all real continuous and bounded functions on I, then we can assert (see for instance [1]):

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THEOREM 1. *For $x \in I$ the following statements are equivalent*:

(a) Z_n^x *converges in distribution to* Z^x $(n \to \infty)$.

(b) $T_n(f, x) \to T(f, x)$ $(n \to \infty)$, *for all* $f \in CB(I)$.

(c) *The same as* (b) *with* $CB(I)$ *replaced by the set of all real uniformly continuous and bounded functions on* I.

In this way problems concerning approximations of positive linear operators can be analyzed in terms of convergence of probability laws.

In the cases most frequently considered in the literature (see for instance [4–6, 8]), Z^x is degenerate at x (i.e., T is the identity operator), the variables Z_n^x are constructed as functions of sums of independent and identically distributed random variables and the desired convergence easily follows from the law of large numbers.

In Theorem 2 below some famous Bernstein-type operators are exhibited as appropriate limits of other ones. This is readily made by using well-known limit theorems of probability theory, in particular the "Poisson approximation to the binomial distribution." Obviously the idea has a wider scope and it can be applied to other situations including multivariate cases. In fact Theorem 1 holds true when I is a metric space.

On the other hand, for the problem (of major interest) of determining the rate of convergence by estimating $|T_n(f, x) - T(f, x)|$, probabilistic methods also work as it can be seen in the references above. Our Theorem 3, which gives rates of convergence for cases (a) and (b) in Theorem 2, is obtained by using results about the rapidity of convergence to the Poisson distribution.

Next we list the operators involved in the mentioned results and introduce some notations.

For $n \in \mathbb{N}$, $x \geqslant 0$, and $f \in CB[0, +\infty)$ define

$$L_n(f, x) = (1+x)^{-n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k$$

which is the operator introduced by Bleimann, Butzer, and Hahn [2]. Similarly,

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} f(k/n) \frac{(nx)^k}{k!}$$

is the Szász operator. The Baskakov operator is defined by

$$B_n^*(f, x) = (1+x)^{-n} \sum_{k=0}^{\infty} f(k/n) \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k,$$

and

$$G_n(f, x) = \begin{cases} \dfrac{x^{-n}}{(n-1)!} \displaystyle\int_0^\infty f(u/n)\, u^{n-1} e^{-u/x}\, du, & \text{if} \quad x > 0 \\ f(0), & \text{if} \quad x = 0 \end{cases}$$

is the Gamma operator.

Finally, for $n \in \mathbb{N}$, $0 \leqslant x \leqslant 1$, and $f \in C[0, 1]$, $B_n(f, x)$ defined by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}$$

is the Bernstein operator.

2. Limiting Properties

Theorem 2. *Let m be a fixed integer and let f be a real continuous and bounded function on the interval $[0, +\infty)$. Then for each $x \geqslant 0$ we have as $n \to \infty$:*

(a) $L_{mn}(f(nu/(1+u)), x/n) \to S_m(f, x)$,

(b) $B_{mn}(f(nu), x/n) \to S_m(f, x)$,

(c) $B^*_{mn}(f(nu), x/n) \to S_m(f, x)$, *and*

(d) $B^*_m(f(u/n), nx) \to G_m(f, x)$.

Proof. To prove (a) observe that

$$L_{mn}\left(f\left(\frac{nu}{1+u}\right), x/n\right) = Ef((m+n^{-1})^{-1}\, U_n^x)$$

and

$$S_m(f, x) = Ef(m^{-1} Z^x),$$

where U_n^x is a random variable with a binomial distribution having parameters mn and $p = p(n) = x(n+x)^{-1}$, and Z^x is a random variable with a Poisson distribution of mean mx. Now, since $mnp(n) \to mx$ (as $n \to \infty$) we have that U_n^x converges in law to Z^x (this is the very classical and famous result about the "Poisson approximation to the binomial distribution"—see for instance [3] or [7]), and therefore it is straightforward to see that $(m+n^{-1})^{-1}\, U_n^x$ converges in law to $m^{-1} Z^x$. So the conclusion follows from Theorem 1.

The same argument is applicable for proving (b). To this end, let $n \geqslant x$ and note that

$$B_{mn}(f(nu), x/n) = Ef(m^{-1}V_n^x),$$

where V_n^x has a binomial distribution with parameters mn and $p = x/n$.

Finally, we deal with (c) and (d). Easy computations show that

$$B_{mn}^*(f(nu), x/n) = Ef(m^{-1}W_n^x)$$

and

$$B_m^*(f(u/n), nx) = Ef(m^{-1}n^{-1}Y_n^x),$$

where W_n^x has a negative binomial distribution of parameters mn and $p = n(n+x)^{-1}$ and Y_n^x has a negative binomial distribution of parameters m and $p = (1+nx)^{-1}$. Now the characteristic functions of W_n^x and $n^{-1}Y_n^x$ are respectively

$$\varphi_n(t) = (1 + n^{-1}x(1 - e^{it}))^{-mn}$$

and

$$\psi_n(t) = (1 + nx(1 - e^{it/n}))^{-m}.$$

Since

$$\varphi_n(t) \to \exp(mx(e^{it} - 1))$$

and

$$\psi_n(t) \to (1 - ixt)^{-m}$$

(as $n \to \infty$), we conclude from the continuity theorem of Lévy that W_n^x (resp. $n^{-1}Y_n^x$) converges in law to a random variable with a Poisson distribution of mean mx (resp. a gamma distribution of parameters $1/x$ and m, if $x > 0$, or the distribution degenerate at 0 in case $x = 0$). Again the conclusions follow from Theorem 1. ∎

Remark. To the best of our knowledge, properties (b), (c), and (d) are observed here for the first time. Property (a) was stated and proved by Rasul A. Khan for f restricted to be uniformly continuous (see [6, Theorem 3]). In view of Theorem 1 above, Theorem 3 in [6] is actually equivalent to our statement but the proof by Khan is wholly different of the one given here. Such a proof is based on certain estimates, concerning the weights of the binomial and Poisson distributions, contained in a previous lemma. In relation to such estimates, more strengthened results are available in the literature on probability theory (see, for instance, [7, p. 345]). We shall use these results in the following section.

3. Rates of Convergence

THEOREM 3. *Let m be a fixed integer and let f be a real continuous and bounded function on* $[0, +\infty)$. *Then we have:*

(a) *For* $n \geqslant x \geqslant 0$,

$$|B_{mn}(f(nu), x/n) - S_m(f, x)| \leqslant \frac{x}{n} 2 \, \|f\| \min(2, mx).$$

(b) *For* $x \geqslant 0$ *and* $n = 1, 2, ...$

$$\left| L_{mn}\left(f\left(\frac{nu}{1+u}\right), x/n \right) - S_m(f, x) \right|$$
$$\leqslant 2\omega(f, x/mn) + (x(1+x)/n) \, 2m \, \|f\| (1 + 2mx) \, e^{2mx},$$

where $\|f\|$ *is the sup-norm of f and* $\omega(f, \delta)$ *is, as usual, the first modulus of continuity of f.*

Proof. In what follows we use the notations used in the proof of Theorem 2.

To prove (a) observe that, for $n \geqslant x \geqslant 0$,

$$|B_{mn}(f(nu), x/n) - S_m(f, x)| \leqslant \sum_{k=0}^{\infty} |f(k/n)| \, |P_k(n) - \pi_k|$$
$$\leqslant \|f\| \sum_{k=0}^{\infty} |P_k(n) - \pi_k|,$$

where, for $k = 0, 1, 2, ...$

$$P_k(n) = P(V_n^x = k) = \binom{mn}{k} (x/n)^k \left(1 - \frac{x}{n}\right)^{mn-k}$$

and

$$\pi_k = P(Z^x = k) = e^{-mx} \frac{(mx)^k}{k!}.$$

Now, from Prohorov's inequality (see [7, p. 345]) we have

$$\sum_{k=0}^{\infty} |P_k(n) - \pi_k| \leqslant \frac{x}{n} 2 \min(2, mx)$$

and the result follows.

To prove (b) note that, for $x \geqslant 0$ and $n = 1, 2, \ldots$, we can write

$$\left| L_{mn}\left(f\left(\frac{nu}{1+u}\right), x/n\right) - S_m(f, x)\right|$$
$$= |Ef((m+n^{-1})^{-1} U_n^x) - Ef(m^{-1}Z^x)|$$
$$\leqslant E\,|f((m+n^{-1})^{-1} U_n^x) - f(m^{-1}U_n^x)|$$
$$+ |Ef(m^{-1}U_n^x) - Ef(m^{-1}Z^x)|. \tag{1}$$

We shall estimate each term on the right in (1) separately. For the first term we use a well known technique (see for instance [5, Theorem 1]). Let $\delta > 0$ and define

$$\lambda = [\delta^{-1}\,|(m+n^{-1})^{-1} U_n^x - m^{-1}U_n^x|]$$
$$= [\delta^{-1}m^{-1}(mn+1)^{-1} U_n^x],$$

where brackets indicate integral part. Then obviously

$$|f((m+n^{-1})^{-1} U_n^x) - f(m^{-1}U_n^x)| \leqslant \omega(f, \delta)(1+\lambda)$$

and therefore

$$E\,|f((m+n^{-1})^{-1} U_n^x) - f(m^{-1}U_n^x)|$$
$$\leqslant \omega(f, \delta)(1 + E\lambda)$$
$$\leqslant \omega(f, \delta)(1 + \delta^{-1}m^{-1}(mn+1)^{-1} EU_n^x)$$
$$= \omega(f, \delta)(1 + nx(n+x)^{-1}(mn+1)^{-1}\delta^{-1})$$
$$\leqslant \omega(f, \delta)(1 + x(mn)^{-1}\delta^{-1}).$$

On taking $\delta = x/mn$ we obtain

$$E\,|f((m+n)^{-1} U_n^x) - f(m^{-1}U_n^x)| \leqslant 2\omega(f, x/mn). \tag{2}$$

Now, for estimating the second term on the right in (1) we proceed as in the proof of (a). In fact

$$|Ef(m^{-1}U_n^x) - Ef(m^{-1}Z^x)| \leqslant \|f\| \sum_{k=0}^{\infty} |P'_k(n) - \pi_k|. \tag{3}$$

where, for $k = 0, 1, 2, \ldots$,

$$P'_k(n) = P(U_n^x = k) = \binom{mn}{k}\left(\frac{x}{n+x}\right)^k \left(\frac{n}{n+x}\right)^{mn-k}.$$

By using the general theorem on page 345 in [7] (instead of Prohorov's inequality which is not applicable here because $mnx(n+x)^{-1} \neq mx$, for $x \neq 0$) we get

$$\sum_{k=0}^{\infty} |P'_k(n) - \pi_k| \leqslant (2 + 4mnx(n+x)^{-1})\, e^{2mx} A(n, x), \tag{4}$$

where

$$A(n, x) = \sup\{|[mnt]\, x(n+x)^{-1} - mxt| : 0 \leqslant t \leqslant 1\},$$

brackets indicating integral part. Since, for $0 \leqslant t \leqslant 1$

$$\begin{aligned} |[mnt]\, x(n+x)^{-1} - mxt| &= |[mnt]\, x - mntx - mtx^2|(n+x)^{-1} \\ &\leqslant (mtx^2 + x)(n+x)^{-1} \\ &\leqslant m(x^2 + x)\, n^{-1}, \end{aligned}$$

we have

$$A(n, x) \leqslant mx(1+x)\, n^{-1}, \tag{5}$$

and so (b) follows from the inequalities (1)–(5). ▮

The next corollary is an inmediate consequence of Theorem 3.

COROLLARY. (a) *For any real continuous and bounded function f on the interval* $[0, +\infty)$ *the convergence*

$$B_{mn}(f(nu), x/n) \to S_m(f, x) \qquad (n \to \infty)$$

is uniform on each bounded interval $[0, a]$.

(b) *For any real uniformly continuous and bounded function f on* $[0, +\infty)$ *the convergence*

$$L_{mn}\left(f\left(\frac{nu}{1+u}\right), x/n\right) \to S_m(f, x) \qquad (n \to \infty)$$

is uniform on each bounded interval $[0, a]$. *Moreover, in both cases, rates of convergence easily follow from Theorem* 3.

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